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LETTER TO THE EDITOR

Integrable spin chain with two- and three-particle interactions

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Abstract. The one-dimensional model which consists of two isotropic XY chains with spins $S=\frac{1}{2}$ coupled by three-spin interactions is considered with the help of the Bethe ansatz. It is shown that the diagonalization of the Hamiltonian can be reduced to solving a set of coupled nonlinear equations. The exact solution of these equations corresponding to the ground state of system is obtained. The model considered exhibits a ground state of Anderson type with the finite magnetization along the quantization axis.

The one-dimensional quantum integrable models were intensively investigated in recent years. The investigation of these models was begun sixty years ago in the well known paper by Bethe [1, 2] concerned with the calculation of the wavefunctions for the one-dimensional Heisenberg magnet with pair interactions and spin $S=\frac{1}{2}$.

The one-dimensional Hubbard model is the other well known example of the integrable lattice system. The exact solution for the wavefunctions and ground state energy of this model was obtained by Lieb and Wu [3]. These authors used the exact solution of the one-dimensional gas of particles with repulsion given by Yang in 1967 [4]. Further development in the theory of integrable models was connected with the generalization of the Bethe ideas and their application to more complicated problems concerned with the study of systems with spin $S>\frac{1}{2}$ [5, 6]. In many cases the models with pair interactions were investigated. Of certain interest is the consideration of Bethe ideas applied to the systems in which both pair and many-particle interactions are present.

In the present letter we consider a spin one-half chain with two- and three-particle interactions. In our case these interactions are competing ones. It leads to the frustrated ground state with the finite magnetization. In this sense the ground state under consideration can be interpreted as the RVB state of Anderson type [7].

The Hamiltonian of the considering model has the following form:

$$H=-\frac{1}{2}\sum_{j=1}^{N}\sum_{\tau=1,2}(\sigma^x_{j(\tau)}\sigma^x_{j+1(\tau)}+\sigma^y_{j(\tau)}\sigma^y_{j+1(\tau)})(1-U\sigma^z_{j+\tau-1(\tau+1)}) \qquad (1)$$

where $\sigma^\alpha_{j(\tau)}(\alpha=x,y,z)$ are Pauli spin matrices of the jth lattice site on the sublattice τ ($\tau=1,2$, $\sigma^\alpha_{j(3)}=\sigma^\alpha_{j(1)}$). We used the periodical boundary condition ($\sigma^\alpha_{N+1(\tau)}=\sigma^\alpha_{1(\tau)}$). The Hamiltonian possesses the obvious symmetry with respect to the change $U\Rightarrow -U$. Therefore in the following we restrict ourselves to the investigation of the case $U>0$. The case $U=0$ corresponds to the model of two non-interacting isotropic XY chains [8]. In the case $U\to\infty$ we obtain the modified XY chain [9]. By using the Jordan–Wigner transformation [8] the Hamiltonian (1) can be presented in terms of the creation and annihilation operators up to the boundary conditions

$$H = -\sum_{j=1}^{N} \sum_{\tau=1,2} (a^{+}_{j(\tau)} a_{j+1(\tau)} + a^{+}_{j+1(\tau)} a_{j(\tau)})(1 - V a^{+}_{j+\tau-1(\tau+1)} a_{j+\tau-1(\tau+1)}) \quad (2)$$

where $V = 2U/(1+U)$.

This Hamiltonian can be interpreted as follows. There is a one-dimensional lattice which consists of two sublattices (see figure 1). The particles move along the sites of each sublattice so that only the jumps between the neighbouring sites of the same sublattice are possible. The interaction between the sublattices means that the energy of the jump between the sites j and $j+1$ depends on whether the site in the other sublattice which corresponds to this pair is occupied or vacant. The presence of such three-site interactions contrasts the given model with the one-dimensional Hubbard model [3] in the Hamiltonian of which the four-fermion interaction is the two-site one. Thus we obtain an interesting problem of statistical mechanics in the formulation of secondary quantization operators as well.

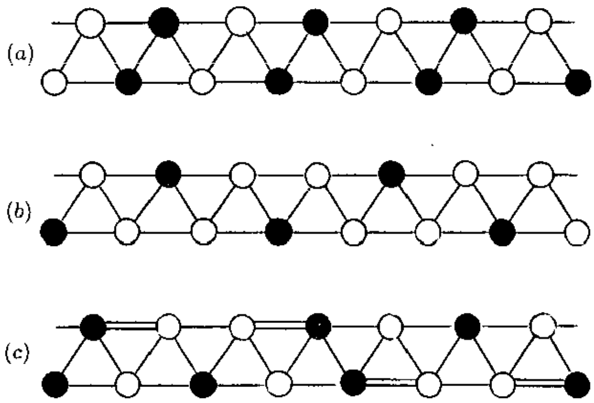

Figure 1. Symmetrical configurations of the ground state for (*a*) $U=0$; (*b*) $U=1$; (*c*) $0<U<1$.

Let us seek the amplitude of the wavefunction of the Hamiltonian (1) which corresponds to the state with m spins directed upward (↑) in the sites with coordinates $x_1 < x_2 < \ldots < x_m$ of the first sublattice and with $(n-m)$ spins ↑ in the sites $x_{m+1} < x_{m+2} < \ldots < x_n$ of the second sublattice in the form of the generalized Bethe ansatz [3, 4, 6, 10]

$$f(x_1, \ldots, x_n) = \sum A^{\tau_{Q_1} \ldots \tau_{Q_n}}_{P_1 \ldots P_n} \prod_{j=1}^{n} \exp\{ik_{P_j}[x_{Q_j} + \tfrac{1}{2}(\tau_{Q_j} - 1)]\}. \quad (3)$$

The summation in this expression is carried out over all permutations $[P_1, \ldots, P_n]$ of the numbers $1, 2, \ldots, n$. The permutation $[Q_1, \ldots, Q_n]$ of the numbers $1, 2, \ldots, n$ is such that $1 \leqslant x_{Q_1} \leqslant x_{Q_2} \leqslant \ldots \leqslant x_{Q_n}$, so that $\tau_{Q_j} = 1, 2$ for $Q_j \leqslant m$ and $Q_j > m$ respectively. The eigenvalue of the Hamiltonian (1) appropriate to this wavefunction is

$$E = -2(1+U) \sum_{j=1}^{n} \cos k_j. \quad (4)$$

The wavefunction which was constructed with the help of coefficients (3) is the eigenfunction of the Hamiltonian (1) if these coefficients satisfy the following conditions

$$A^{\ldots\tau_1\tau_2\ldots}_{\ldots P_1P_2\ldots} = \sum_{\tau,\tau'=1,2} S^{\tau'\tau}_{\tau_1\tau_2}(k_{P_1}-k_{P_2})A^{\ldots\tau\tau'\ldots}_{\ldots P_2P_1\ldots} \tag{5}$$

$$A^{\tau_1\ldots\tau_n}_{P_1\ldots P_n} = A^{\tau_2\ldots\tau_n\tau_1}_{P_2\ldots P_nP_1}\exp(ik_{P_1}N) \tag{6}$$

where the non-vanishing elements of S-matrix are

$$S^{11}_{11}(k) = S^{22}_{22}(k) = -1 \qquad S^{12}_{12}(k) = S^{21}_{21}(k) = \sin\tfrac{1}{2}k/\sin(\tfrac{1}{2}k - i\alpha)$$

$$S^{12}_{21}(k) = S^{21}_{12}(k) = i\,\mathrm{sh}\,\alpha/\sin(\tfrac{1}{2}k - i\alpha) \qquad \mathrm{e}^{\alpha} = (1+U)/(1-U).$$

A necessary and sufficient condition for the compatibility of equation (5) is the fulfilment of the Yang–Baxter relations [4, 5]. In our case the S-matrix satisfies these relations and we may use the quantum method of the inverse problem [11] to solve equations (5), (6). As result, we have

$$\exp(ik_jN) = (-1)^{n-m+1}\prod_{\beta=1}^{m}\frac{\sin[\frac{1}{2}(k_j-\Lambda_\beta)+i\alpha']}{\sin[\frac{1}{2}(k_j-\Lambda_\beta)-i\alpha']} \qquad \alpha' = \alpha/2$$

$$(-1)^n\prod_{j=1}^{n}\frac{\sin[\frac{1}{2}(\Lambda_\beta-k_j)+i\alpha']}{\sin[\frac{1}{2}(\Lambda_\beta-k_j)-i\alpha']} = \prod_{\substack{\gamma=1\\ \gamma=\beta}}^{m}\frac{\sin[\frac{1}{2}(\Lambda_\beta-\Lambda_\gamma)+2i\alpha']}{\sin[\frac{1}{2}(\Lambda_\beta-k_j)-2i\alpha']}.$$

Taking the logarithm of both sides of these equations, we obtain

$$k_jN + \sum_{\beta=1}^{m}\theta(k_j-\Lambda_\beta,\alpha') = 2\pi I_j \qquad (j=1,2,\ldots,n)$$

$$\sum_{j=1}^{n}\theta(\Lambda_\beta-k_j,\alpha') - \sum_{\gamma=1}^{m}\theta(\Lambda_\beta-\Lambda_\gamma,2\alpha') = 2\pi J_\beta \qquad (\beta=1,2,\ldots,m)$$

$$\theta(k,\alpha') = 2\arctan[\coth(\alpha')\tanh\tfrac{1}{2}k] \qquad -\pi\leqslant\theta(k,\alpha)\leqslant\pi \tag{7}$$

where I_j and J_β are half-integer (integer) numbers for even (odd) n and m, respectively.

Thus, we have obtained the energy spectrum and eigen-vectors of the Hamiltonian under consideration through the solution of the system of equations (7). In the present letter we restrict ourselves with the investigation of the ground state, which corresponds to the following values of I_j and J_β

$$I_j = j-(n+1)/2 \qquad (j=1,2,\ldots,n)$$

$$J_\beta = \beta-(m+1)/2 \qquad (\beta=1,2,\ldots,m)$$

A momentum of this state is

$$P = \sum_{j=i}^{n}k_j = 2\pi\left(\sum_{j=i}^{n}I_j + \sum_{\beta=i}^{m}J_\beta\right)N^{-1} = 0.$$

Following the pioneering works on Bethe ansatz [5, 6], we assume that in the thermodynamical limit $N\to\infty$, $n\to\infty$, $m\to\infty$ for fixed ratios n/N and m/N the values of k_j and Λ_β fill the intervals $[-Q, Q]$ and $[-B, B]$ uniformly with densities $\rho(k)$ and $\sigma(\Lambda)$, respectively. Then, instead of (7), we obtain the system of integral equations

$$
\begin{aligned}
&2\pi\rho(k)=1+\int_{-B}^{B}\theta'(k-\Lambda;\,\alpha')\sigma(\Lambda)\,\mathrm{d}\Lambda \qquad \theta'(k;\,\alpha')=\sinh\alpha(\cosh\alpha-\cos k)^{-1}\\
&2\pi\sigma(\Lambda)=\int_{-Q}^{Q}\theta'(\Lambda-k;\,\alpha')\rho(k)\,\mathrm{d}k-\int_{-B}^{B}\theta'(\Lambda-\Lambda';\,2\alpha')\sigma(\Lambda')\,\mathrm{d}\Lambda' \qquad (8)\\
&\int_{-Q}^{Q}\rho(k)\,\mathrm{d}k=n/N \qquad \int_{-B}^{B}\sigma(\Lambda)\,\mathrm{d}\Lambda=m/N.
\end{aligned}
$$

Similarly, going over to the continuous distribution in (4) we obtain

$$E=-2N(1+U)\int_{-Q}^{Q}\cos k\rho(k)\,\mathrm{d}k. \qquad (9)$$

From the symmetry of the system it is clear that in the ground state $m=n/2$, and it corresponds to $B=\pi$. Then, excluding the function $\sigma(\Lambda)$ equations (8), we obtain

$$
\begin{aligned}
&2\pi\rho(k)-\int_{-Q}^{Q}\varphi(k-k')\rho(k')\,\mathrm{d}k'=1 \qquad y=1-\int_{-Q}^{Q}\rho(k)\,\mathrm{d}k\\
&\varphi(k)=\tfrac{1}{2}+2\sum_{n=1}^{\infty}\cos(nk)/[1+\exp(2|\alpha|n)].
\end{aligned} \qquad (10)
$$

These equations determine the ground state energy (9) as a function of the magnetization y. The analysis of (10) shows that the function $E(y)$ has a wonderful peculiarity, namely, this function has an absolute minimum for $y=0$, in this case $Q=\pi/2$

$$y_0=1-\int_{-\pi/2}^{\pi/2}\rho_0(k)\,\mathrm{d}k \qquad \frac{1}{N}E_0=-2(1+U)\int_{-\pi/2}^{\pi/2}\rho_0(k)\cos k\,\mathrm{d}k.$$

The function $\rho_0(k)$ satisfies equations (10) at $Q=\pi/2$. The solution of this equation can be obtained with the help of numerical integration or using the perturbation theory. For example, at large α we have

$$
\begin{aligned}
&y_0=\frac{1}{3}-\frac{32}{9\pi^2}\mathrm{e}^{-2|\alpha|}+\mathrm{O}(\mathrm{e}^{-4|\alpha|})\\
&\frac{1}{N}E_0=-2(1+U)\left[\frac{4}{3\pi}+\frac{2}{3\pi}\left(1+\frac{8}{3\pi}\right)\mathrm{e}^{-2|\alpha|}+\mathrm{O}(\mathrm{e}^{-4|\alpha|})\right].
\end{aligned}
$$

The presence of this minimum shows that in the ground state in the zero field the considering systems has a finite magnetization along the z axis. The value of this magnetization is determined by the value of the interaction constant U (or V for the model (2)).

Let us interpret the obtained result using Anderson's picture of the ground state, the so-called resonating-valence-bond (RVB) state [7]. For the vanishing interaction $\alpha=0$ ($U=0$) in the ground state we have equal numbers of spins ↑ and ↓. The symmetrical configuration is presented in figure 1(*a*). This ground state can be envisaged as a linear combination of wavefunctions built up of singlet pairs (SP). We may think of it as a liquid-type state in which the system of pair bonds is resonating between the possible configurations. In the other limiting case $\alpha\to\infty$ ($U\to 1$) the number of spins ↑ is two times smaller than the number of spins ↓ (see figure 1(*b*)). The ground state is a superposition of states of triangles in which there are one spin ↑ and two spins ↓, i.e. these triangles (MT) have a magnetic momentum. In the general case ($0 < U < 1$) the ground state is the linear combination of SP and MT. In the symmetrical configuration MT is surrounded by an interchangeable pair (IP); in figure 1(*c*) IP is shown by a double line. States of spins in this pair can be interchanged without breaking short-range order. The interchanging of the spin states in IP leads to the moving of MT a distance of two lattice constants. The MT moves along the lattice with the transfer of the magnetic momentum without change of the energy of system. Thus the ground state may be presented as a magnetic liquid or as a conducting liquid if we consider the Hamiltonian (2).

Of course, this interpretation is preliminary and naive. For more detailed understanding of the model it is necessary to calculate the excitation spectrum and it will be the subject of a subsequent investigation.

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